AD/AP-24

Accelerator Development Department
Accelerator Physics Division
BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, NY 11973

Accelerator Physics Technical Note No. 24

Beta-Function Distortions Due to Linear Coupling

V. Garczynski

August 1991

BETA-FUNCTION DISTORTIONS DUE TO LINEAR COUPLING

V. Garczynski
Accelerator Development Department
Brookhaven National Laboratory
August 1991

1. Introduction

Since a coupling between transverse x and y degrees of freedom is expected in RHIC it is important to examine its influence on beta-functions.[1] We shall calculate the shifts of the beta-functions produced by point skew-quadrupoles distributed around the ring. The X-Y coupling is linear in this case[2] and its effect can be calculated exactly assuming that k-th skew-quadrupole of length ℓ_k is located at s_k in the ring and has strength q_k.

$$q_k = (\beta_x \beta_y)^{1/2} \frac{\ell_k}{\rho} a_{1_{|s=s_k}}, \quad k = 1, \dots, N \tag{1.1}$$

where β_x, β_y are beta-functions of a perfect machine.[3]

It is known that in the presence of linear coupling there exists a matrix R such that in passing to new variables u, u', v, v' the transverse motions are decoupled i.e.,

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = R \begin{bmatrix} u \\ u' \\ v \\ v' \end{bmatrix}, \tag{1.2}$$

$$T = \begin{bmatrix} M & m \\ m & N \end{bmatrix} = R \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} R^{-1}. \tag{1.3}$$

Here T is a single turn 4×4 symplectic transfer matrix for a coupled machine and A, B are symplectic 2×2 submatrices describing uncoupled transverse motions

$$A = \begin{bmatrix} \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \\ -\gamma_1 \sin\mu_1 & \cos\mu_1 - \alpha_1 \sin\mu_1 \end{bmatrix}, \tag{1.4}$$

$$B = \begin{bmatrix} \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \\ -\gamma_2 \sin\mu_2 & \cos\mu_2 - \alpha_2 \sin\mu_2 \end{bmatrix}, \tag{1.5}$$

α_k, β_k, γ_k, μ_k, $k = 1, 2$ are the usual Courant-Snyder parameters of decoupled motions.

We would like to calculate the beta-function distortions

$$\Delta\beta_x = \beta_1 - \beta_x, \tag{1.6}$$

$$\Delta B_y = \beta_2 - \beta_y, \tag{1.7}$$

assuming that the linear coupling is small. This can be done using general formulae that express the A, B submatrices in terms of the T matrix

$$A = M + (t + \delta)^{-1} (\overline{m} + m) m, \tag{1.8}$$

$$B = N - (t + \delta)^{-1} (m + \overline{n}) n, \tag{1.9}$$

where

$$t = \frac{1}{2} Tr (M - N), \tag{1.10}$$

and

$$\delta = \left(t^2 + |\overline{m} + n|\right)^{1/2} . \tag{1.11}$$

Here $\overline{m}$ stands for a symplectic conjugate of m

$$m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \to \overline{m} = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} . \tag{1.12}$$

The single turn transfer matrix T can be written as a polynominal

$$T = \sum_{k=0}^{N} T^{(k)}, \tag{1.13}$$

where $T^{(k)}$ are given via, k-th order in the q's, driving terms. In this note we shall calculate the beta-function distortions up to the second order in the q's.

2. Calculations of the Beta-Function Distortions

According to the formula (1.8), (1.9) one has the relations

$$A_{12} = \beta_1 \sin\mu_1 = M_{12} + (t+\delta)^{-1}\left[(\overline{m}+n)\, m\right]_{12}, \tag{2.1}$$

$$B_{12} = \beta_2 \sin\mu_2 = N_{12} - (t+\delta)^{-1}\left[(m+\overline{n})\, m\right]_{12}. \tag{2.2}$$

Taking into account the tune splitting

$$\mu_1 = \mu_x + \Delta\mu_1, \tag{2.3}$$

$$\mu_2 = \mu_y + \Delta\mu_2, \tag{2.4}$$

where $\Delta\mu_1$, $\Delta\mu_2$ are expressed through second-order driving terms, one gets from the expansion (1.13), assuming $\alpha_x(0) = \alpha_y(0) = 0$, the results

$$\frac{\Delta\beta_x}{\beta_x} = \frac{1}{2}\left(d_{cs}^{(2)} - d_{sc}^{(2)}\right)\cot^2\mu_x + \left[\frac{1}{2}\left(d_{cc}^{(2)} + d_{ss}^{(2)}\right) - d_{ss}^{(2)}\right]\cot\mu_x +$$
$$+ d_{cs}^{(2)} + (t+\delta)^{-1}\left\{-\left[\left(d_{sc}^{(1)}\right)^2 + \left(d_{ss}^{(1)}\right)^2\right]\sin\mu_y\cot\mu_x - \left(d_{sc}^{(2)} - d_{cs}^{(2)}\right)\cos\mu_y + \right.$$
$$\left. + \left(d_{cc}^{(1)} d_{sc}^{(1)} + d_{ss}^{(1)} d_{cs}^{(1)}\right)\sin\mu_y\right\} + O\left(q^4\right), \tag{2.5}$$

$$\frac{\Delta\beta_y}{\beta_y} = \left.\frac{\Delta\beta_x}{\beta_x}\right|_{x\leftrightarrow y} \tag{2.6}$$

Here the driving terms of the first order $d^{(1)}$ and of the second order $d^{(2)}$ are defined as follows:

$$\begin{bmatrix} d_{ss}^{(1)} \\ d_{sc}^{(1)} \\ d_{cs}^{(1)} \\ d_{cc}^{(1)} \end{bmatrix} = \sum_{r=1}^{N} q_r \begin{bmatrix} \sin\mu_x^r \sin\mu_y^r \\ \sin\mu_x^r \cos\mu_y^r \\ \cos\mu_x^r \sin\mu_y^r \\ \cos\mu_x^r \cos\mu_y^2 \end{bmatrix}, \tag{2.7}$$

and

$$\begin{bmatrix} d_{ss}^{(2)} \\ d_{sc}^{(2)} \\ d_{cs}^{(2)} \\ d_{cc}^{(2)} \end{bmatrix} = \sum_{1\le r<s\le N} q_r q_s \sin\left(\mu_y^s - \mu_y^r\right) \begin{bmatrix} \sin\mu_x^s \sin\mu_x^r \\ \sin\mu_x^s \cos\mu_x^r \\ \cos\mu_x^s \sin\mu_x^r \\ \cos\mu_x^s \cos\mu_x^r \end{bmatrix}, \tag{2.8}$$

and μ_x^r, μ_y^r are phase advances

$$\mu_x^r = \int_0^{S_r} \frac{ds}{\beta_x} \quad , \quad \mu_y^r \int_0^{S_r} \frac{ds}{\beta_y}. \tag{2.9}$$

Additional sets of driving terms, denoted as $\check{d}_{ss}^{(1)}$, $\check{d}_{ss}^{(2)}$ etc, are obtained from the above equations by simply exchanging x and y,

$$\check{d}^{(k)}(x,y) = d^{(k)}(y,x), k = 1,2 \tag{2.10}$$

It is easy to notice the relations

$$\begin{aligned}
\check{d}_{ss}^{(1)} &= d_{ss}^{(1)}, \\
\check{d}_{sc}^{(1)} &= d_{cs}^{(1)}, \\
\check{d}_{cs}^{(1)} &= d_{sc}^{(1)}, \\
\check{d}_{cc}^{(1)} &= d_{cc}^{(1)}.
\end{aligned} \tag{2.11}$$

It is interesting to check if the beta-function distortions disappear after correction of the tune-splitting which requires, among others, that the following equalities hold

$$\begin{aligned}
d_{cc}^{(2)} - d_{ss}^{(2)} &= 0, \\
\check{d}_{cc}^{(2)} - \check{d}_{ss}^{(2)} &= 0, \\
d_{cs}^{(2)} - d_{sc}^{(2)} &= 0, \\
\check{d}_{cs}^{(2)} - \check{d}_{sc}^{(2)} &= 0.
\end{aligned} \tag{2.12}$$

Applying them to the formula (2.5), (2.6) one finds that residual beta-function distortions are present

$$\begin{aligned}
\frac{\Delta\beta_x}{\beta_x} = -d_{ss}^{(2)} \cot\mu_x + d_{cs}^{(2)} + (t+\delta)^{-1}_{|\Delta\nu=0} \Big\{ - \Big[\left(d_{sc}^{(1)}\right)^2 + \left(d_{ss}^{(1)}\right)^2 \Big] \sin\mu_y \cot\mu_x + \\
+ \left(d_{cc}^{(1)} d_{sc}^{(1)} + d_{ss}^{(1)} d_{cs}^{(1)} \right) \sin\mu_y \Big\} + O\left(q^4\right),
\end{aligned} \tag{2.13}$$

$$\frac{\Delta\beta_y}{\beta_y}_{|\Delta\nu=0} = \frac{\Delta\beta_x}{\beta_x}_{|\Delta\nu=0, x\leftrightarrow y}, \tag{2.14}$$

and, according to the formula (1.10), (1.11)

$$t+\delta_{|_{\Delta\nu=0}} = 2\left(\cos\mu_x - \cos\mu_y\right) + O\left(q^4\right). \tag{2.15}$$

One sees that passing to the limit $\nu_x - \nu_y \to 0$ is delicate here, and higher order terms in the last expansion should be included. This is essential since RHIC is designed to operate at almost equal tunes: $\nu_x = 28.826$, $\nu_y = 28.821$.

Our results should be compared with perturbative calculations of the beta-function distortions.[4,5] Clearly, the residual beta-function distortions can be removed if we correct driving terms which appear in the formula (2.13), (2.14).

Assuming that skew-quadrupole errors are randomly distributed around the ring and taking into account that

$$N < q^2 >= G_0^2 \tag{2.16}$$

where, for RHIC we take

$$G_0 \simeq 0.25, \tag{2.17}$$

we get the estimate for the average distortion

$$< \frac{\Delta\beta_x}{\beta_x} >\simeq -0.25, \tag{2.18}$$

and similar for $< \frac{\Delta\beta_y}{\beta_y} >$.

Even larger beta-function distortions are expected in SSC in which case $G_0 \simeq 0.5$ which yields for the average beta-function distortion ≈ -1.2.

3. References

1. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).
2. D.A. Edwards and L.C. Teng, IEEE Trans. Nucl. Sci., Vol. NS-20, No. 3, p. 885 (1973).
3. A.G. Ruggiero, "The Problem of Linear Coupling, Parts I, II, III", Talks at Accelerator Physics Division Meetings, May 16, 23 and 30 (1991), BNL.
4. W.E. Gabella, R.D. Ruth and R.L. Warnock, Second Advanced ICFA Beam Dynamics Workshop, Lugano, 1988, CERN 88-04.
5. G. Parzen, BNL Report, AD/RHIC-102, July 1991, and references to earlier work on the subject given there.